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# Weighted Imbedding Theorems in the Space of Differential Forms

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We prove $A_r(\Omega)$-weighted imbedding theorems for differential forms. These results can be used to study the weighted norms of the homotopy operator $T$ from the Banach space $L^p(D, \wedge^l)$ to the Sobolev space $W^{1,p}(D, \wedge^{l-1})$, $l = 1, 2, \ldots, n$, and to establish the basic weighted $L^p$-estimates for differential forms. © 2001 Academic Press

## 1. INTRODUCTION

In recent years many results about Sobolev functions have been extended to differential forms in $\mathbf{R}^n$. The imbedding theorems play crucial roles in generalizing these results to differential forms. The objective of this paper is to prove the weighted versions of imbedding theorems for differential forms and establish weighted norm estimates for the homotopy operator $T$. Let $e_1, e_2, \ldots, e_n$ be the standard unit basis of $\mathbf{R}^n$, $n \geq 2$, and let $\wedge^l = \wedge^l(\mathbf{R}^n)$ be the linear space of $l$-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered $l$-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$, $l = 0, 1, \ldots, n$. We write $\mathbf{R} = \mathbf{R}^1$. The Grassman algebra $\wedge = \oplus \wedge^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all $l$-tuples $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. The Hodge star operator $\star\colon \wedge \to \wedge$ is defined by the rule $\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ and $\alpha \wedge \star\beta = \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1)$ for all $\alpha, \beta \in \wedge$. The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \wedge^0 = \mathbf{R}$. The Hodge star is an isometric isomorphism on $\wedge$ with $\star\colon \wedge^l \to \wedge^{n-l}$ and $\star\star (-1)^{l(n-l)}\colon \wedge^l \to \wedge^l$.



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We always assume that $\Omega$ is a bounded domain in $\mathbf{R}^n$ throughout this paper. Balls are denoted by $B$, and $\sigma B$ is the ball with the same center as $B$ and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We do not distinguish the balls from cubes throughout this paper. The $n$-dimensional Lebesgue measure of a set $E \subseteq \mathbf{R}^n$ is denoted by $|E|$. We call $w(x)$ a weight if $w \in L^1_{\text{loc}}(\mathbf{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$ and a weight $w(x)$, we denote the weighted $L^p$-norm of a measurable function $f$ over $E$ by

$$||f||_{p, E, w^\alpha} = \left( \int_E |f(x)|^p w^\alpha dx \right)^{1/p},$$

where $\alpha$ is a real number.

A differential $l$-form $\omega$ on $\Omega$ is a de Rham current (see [10, Chap. III]) on $\Omega$ with values in $\wedge^l(\mathbf{R}^n)$. We use $D'(\Omega, \wedge^l)$ to denote the space of all differential $l$-forms and $L^p(\Omega, \wedge^l)$ to denote the $l$-forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$ with $\omega_I \in L^p(\Omega, \mathbf{R})$ for all ordered $l$-tuples $I$. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$||\omega||_{p, \Omega} = \left( \int_\Omega |\omega(x)|^p dx \right)^{1/p} = \left( \int_\Omega \left( \sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.$$

For $\omega \in D'(\Omega, \wedge^l)$ the vector-valued differential form

$$\nabla \omega = \left( \frac{\partial \omega}{\partial x_1}, \ldots, \frac{\partial \omega}{\partial x_n} \right)$$

consists of differential forms,

$$\frac{\partial \omega}{\partial x_i} \in D'(\Omega, \wedge^l),$$

where the partial differentiation is applied to the coefficients of $\omega$.

As usual, $W^{1,p}(\Omega, \wedge^l)$ is used to denote the Sobolev space of $l$-forms, which equals $L^p(\Omega, \wedge^l) \cap L^p_1(\Omega, \wedge^l)$ with norm

$$\|\omega\|_{W^{1,p}(\Omega)} = \|\omega\|_{W^{1,p}(\Omega, \wedge^l)} = \operatorname{diam}(\Omega)^{-1} \|\omega\|_{p, \Omega} + \|\nabla \omega\|_{p, \Omega}.$$

The notations $W^{1,p}_{\text{loc}}(\Omega, \mathbf{R})$ and $W^{1,p}_{\text{loc}}(\Omega, \wedge^l)$ are self-explanatory. For $0 < p < \infty$ and a weight $w(x)$, the weighted norm of $\omega \in W^{1,p}(\Omega, \wedge^l)$ over $\Omega$ is denoted by

$$\begin{aligned} \|\omega\|_{W^{1,p}(\Omega), w^\alpha} &= \|\omega\|_{W^{1,p}(\Omega, \wedge^l), w^\alpha} \\ &= \operatorname{diam}(\Omega)^{-1} \|\omega\|_{p, \Omega, w^\alpha} + \|\nabla \omega\|_{p, \Omega, w^\alpha}, \end{aligned} \tag{1.1}$$

where $\alpha$ is a real number.

We denote the exterior derivative by $d$: $D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0, 1, \dots, n$. Its formal adjoint operator $d^\star$: $D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^\star = (-1)^{nl+1} \star d \star$ on $D'(\Omega, \wedge^{l+1})$, $l = 0, 1, \dots, n$.

Iwaniec and Lutoborski prove the following result in [7]. Let $D \subset \mathbf{R}^n$ be a bounded, convex domain. To each $y \in D$ there corresponds a linear operator $K_y$: $C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1, \dots, \xi_l) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \dots, \xi_{l-1})\, dt$$

and the decomposition

$$\omega = d(K_y \omega) + K_y(d\omega).$$

A homotopy operator $T$: $C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by averaging $K_y$ over all points $y$ in $D$,

$$T\omega = \int_D \varphi(y) K_y \omega\, dy, \tag{1.2}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y) dy = 1$. By substituting $z = tx + y - ty$, (1.2) reduces to

$$T\omega(x, \xi) = \int_D \omega(z, \zeta(z, x - z), \xi)\, dz, \tag{1.3}$$

where the vector function $\zeta$: $D \times \mathbf{R}^n \to \mathbf{R}^n$ is given by

$$\zeta(z, h) = h \int_0^\infty s^{l-1} (1 + s)^{n-1} \varphi(z - sh)\, ds.$$

Proposition 4.1 in [7] says that the integral (1.3) defines a bounded operator

$$T\colon L^s(D, \wedge^l) \to W^{1,s}(D, \wedge^{l-1}), \qquad l = 1, 2, \dots, n,$$

with norm estimated by

$$\|Tu\|_{W^{1,s}(D)} \le C|D| \|u\|_{s,D}.$$

Many interesting results of differential forms and their applications in fields such as potential theory and the theory of elasticity have been found (see [1–4, 7–9]. For many purposes, we need to know the integrability of differential forms and estimate the integrals for differential forms. In this paper, we are going to develop some results which can be used to study the integrability of differential forms and estimate the integrals for differential forms. Specifically, we are going to prove the $A_r(\Omega)$-weighted imbedding theorems for differential forms satisfying the $A$-harmonic equation

$$d^\star A(x, d\omega) = 0, \tag{1.4}$$

where $A: \Omega \times \wedge^l (\mathbf{R}^n) \to \wedge^l (\mathbf{R}^n)$ satisfies the conditions

$$|A(x, \xi)| \leq a|\xi|^{p-1} \tag{1.5}$$

and

$$\langle A(x, \xi), \xi \rangle \ \geq \ |\xi|^p \tag{1.6}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l (\mathbf{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (1.4). A solution to (1.4) is an element of the Sobolev space $W^{1,p}_{\rm loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi \rangle = 0$$

for all $\varphi \in W^{1,p}(\Omega, \wedge^{l-1})$ with compact support.

A differential $l$-form $u \in D'(\Omega, \wedge^l)$ is called a closed form if $du = 0$ in $\Omega$. Similarly, a differential $l+1$-form $v \in D'(\Omega, \wedge^{l+1})$ is called a coclosed form if $d^\star v = 0$. Clearly, the $A$-harmonic equation is not affected by adding a closed form to $\omega$. Therefore, any type of estimate about $u$ must be modulo a closed form.

## 2. THE LOCAL WEIGHTED IMBEDDING THEOREMS

DEFINITION 2.1. A weight $w(x)$ is called an $A_r$-weight for some $r > 1$ in a domain $\Omega$, written $w \in A_r(\Omega)$, if $w(x) > 0$ a.e., and

$$\sup_B \left( \frac{1}{|B|} \int_B w \, dx \right) \left( \frac{1}{|B|} \int_B \left( \frac{1}{w} \right)^{1/(r-1)} dx \right)^{(r-1)} < \infty$$

for any ball $B \subset \Omega$.

See [5] and [6] for properties of $A_r(\Omega)$-weights. We will need the following generalized Hölder inequality.

LEMMA 2.2. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$, and $s^{-1} = \alpha^{-1} + \beta^{-1}$. If $f$ and $g$ are measurable functions on $\mathbf{R}^n$, then*

$$\|fg\|_{s,\Omega} \ \leq \ \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$$

*for any $\Omega \subset \mathbf{R}^n$.*

From results appearing in [7], we have the following lemma.

LEMMA 2.3. *Let $u \in L^s_{\rm loc}(B, \wedge^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a differential form in a ball $B \subset \mathbf{R}^n$.*

$$\|\nabla(Tu)\|_{s,B} \leq C|B| \|u\|_{s,B}, \tag{2.4}$$

$$\|Tu\|_{s,B} \leq C|B| \operatorname{diam}(B) \|u\|_{s,B}. \tag{2.5}$$

We also need the following lemma [5].

LEMMA 2.6. *If $w \in A_r(\Omega)$, then there exist constants $\beta > 1$ and $C$, independent of $w$, such that*

$$\| w \|_{\beta, B} \le C|B|^{(1-\beta)/\beta} \| w \|_{1, B}$$

*for all balls $B \subset \mathbf{R}^n$.*

The following weak reverse Hölder inequality appears in [9].

LEMMA 2.7. *Let $u$ be an $A$-harmonic tensor in $\Omega$, $\rho > 1$, and $0 < s, t < \infty$. Then there exists a constant $C$, independent of $u$, such that*

$$\|u\|_{s, B} \le C|B|^{(t-s)/st} \|u\|_{t, \rho B}$$

*for all balls or cubes $B$ with $\rho B \subset \Omega$.*

THEOREM 2.8. *Let $u \in L^s_{\text{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying (1.4) in a bounded domain $\Omega \subset \mathbf{R}^n$ and let $T\colon C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator defined in (1.2). Assume that $\rho > 1$ and $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$, such that*

$$\left( \int_B |\nabla(Tu)|^s w^\alpha dx \right)^{1/s} \le C|B| \left( \int_{\rho B} |u|^s w^\alpha dx \right)^{1/s} \tag{2.9}$$

*for any real number $\alpha$ with $0 < \alpha \le 1$.*

Note that (2.9) can be written as

$$\|\nabla(Tu)\|_{s, B, w^\alpha} \le C|B| \|u\|_{s, \rho B, w^\alpha}. \tag{2.9$'$}$$

*Proof.* First, we prove that (2.9) is true for $0 < \alpha < 1$. Let $t = s/(1-\alpha)$. Using Lemma 2.2 yields

$$\begin{aligned} \left( \int_B |\nabla(Tu)|^s w^\alpha dx \right)^{1/s} &= \left( \int_B (|\nabla(Tu)| w^{\alpha/s})^s dx \right)^{1/s} \\ &\le \|\nabla(Tu)\|_{t, B} \left( \int_B w^{t\alpha/(t-s)} dx \right)^{(t-s)/st} \\ &= \|\nabla(Tu)\|_{t,B} \left( \int_B w \, dx \right)^{\alpha/s}. \end{aligned} \tag{2.10}$$

From Lemma 2.3, we obtain

$$\|\nabla(Tu)\|_{t, B} \le C_1 |B| \|u\|_{t, B}. \tag{2.11}$$

Choosing $m = s/(1+\alpha(r-1))$, we have $m < s$. Substituting (2.11) into (2.10) and using Lemma 2.7, we find that

$$\begin{aligned}\left(\int_B |\nabla(Tu)|^s w^\alpha dx\right)^{1/s} &\le C_1|B|\|u\|_{t,B}\left(\int_B w\,dx\right)^{\alpha/s} \\ &\le C_2|B||B|^{(m-t)/mt}\|u\|_{m,\rho B}\left(\int_B w\,dx\right)^{\alpha/s}. \quad (2.12)\end{aligned}$$

Using Lemma 2.2 with $1/m = 1/s + (s-m)/sm$, we have

$$\begin{aligned}\|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m dx\right)^{1/m} \\ &= \left(\int_{\rho B} (|u| w^{\alpha/s} w^{-\alpha/s})^m dx\right)^{1/m} \quad (2.13) \\ &\le \left(\int_{\rho B} |u|^s w^\alpha dx\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}\end{aligned}$$

for all balls $B$ with $\rho B \subset \Omega$. Substituting (2.13) into (2.12), we obtain

$$\begin{aligned}&\left(\int_B |\nabla(Tu)|^s w^\alpha dx\right)^{1/s} \\ &\quad\le C_2|B||B|^{(m-t)/mt}\left(\int_{\rho B} |u|^s w^\alpha dx\right)^{1/s} \quad (2.14) \\ &\qquad\times\left(\int_B w\,dx\right)^{\alpha/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{\alpha(r-1)/s}.\end{aligned}$$

Since $w \in A_r(\Omega)$, we find that

$$\begin{aligned}\|w\|_{1,B}^{\alpha/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{\alpha/s} &\le \left(\left(\int_{\rho B} w\,dx\right)\left(\int_{\rho B}(1/w)^{1/(r-1)}dx\right)^{r-1}\right)^{\alpha/s} \\ &= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w\,dx\right)\right. \quad (2.15) \\ &\qquad\left.\times\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{r-1}\right)^{\alpha/s} \\ &\le C_3|B|^{\alpha r/s}.\end{aligned}$$

Combining (2.15) and (2.14), we find that

$$\left(\int_B |\nabla(Tu)|^s w^\alpha dx\right)^{1/s} \le C_4|B|\left(\int_{\rho B} |u|^s w^\alpha dx\right)^{1/s} \quad (2.16)$$

for all balls $B$ with $\rho B \subset \Omega$. We have proved that (2.9) is true if $0 < \alpha < 1$.

Next, we show (2.9) is also true for $\alpha = 1$, that is, we need to prove that

$$\|\nabla(Tu)\|_{s,B,w} \le C|B|\|u\|_{s,\rho B,w}. \tag{2.17}$$

By Lemma 2.6, there exist constants $\beta > 1$ and $C_5 > 0$, such that

$$\|w\|_{\beta,B} \le C_5|B|^{(1-\beta)/\beta}\|w\|_{1,B} \tag{2.18}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta-1)$; then $1 < s < t$ and $\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/st$, by Lemma 2.2, (2.4), and (2.18), we have

$$\begin{aligned}
\left(\int_B |\nabla(Tu)|^s w\,dx\right)^{1/s} &= \left(\int_B (|\nabla(Tu)|w^{1/s})^s dx\right)^{1/s} \\
&\le \left(\int_B |\nabla(Tu)|^t dx\right)^{1/t}\left(\int_B (w^{1/s})^{st/(t-s)} dx\right)^{(t-s)/st} \\
&\le C_6\|\nabla(Tu)\|_{t,B}\cdot\|w\|_{\beta,B}^{1/s} \qquad (2.19)\\
&\le C_6|B|\|u\|_{t,B}\cdot\|w\|_{\beta,B}^{1/s} \\
&\le C_7|B||B|^{(1-\beta)/\beta s}\|w\|_{1,B}^{1/s}\cdot\|u\|_{t,B} \\
&\le C_7|B||B|^{-1/t}\|w\|_{1,B}^{1/s}\cdot\|u\|_{t,B}.
\end{aligned}$$

Choosing $m = s/r$, from Lemma 2.7, we have

$$\|u\|_{t,B} \le C_8|B|^{(m-t)/mt}\|u\|_{m,\rho B}. \tag{2.20}$$

Lemma 2.2 yields

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} (|u|w^{1/s}w^{-1/s})^m dx\right)^{1/m} \\
&\le \left(\int_{\rho B} |u|^s w\,dx\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)} dx\right)^{(r-1)/s} \qquad (2.21)
\end{aligned}$$

for all balls $B$ with $\rho B \subset \Omega$. Note that $w \in A_r(\Omega)$. Then

$$\begin{aligned}
\|w\|_{1,B}^{1/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{1/s} &\le \left(\left(\int_{\rho B} w\,dx\right)\left(\int_{\rho B}(1/w)^{1/(r-1)}dx\right)^{r-1}\right)^{1/s} \\
&= \left(|\rho B|^r\left(\frac{1}{|\rho B|}\int_{\rho B} w\,dx\right)\right. \qquad (2.22)\\
&\quad \left.\times\left(\frac{1}{|\rho B|}\int_{\rho B}\left(\frac{1}{w}\right)^{1/(r-1)}dx\right)^{r-1}\right)^{1/s} \\
&\le C_9|B|^{r/s}.
\end{aligned}$$

Combining (2.19), (2.20), (2.21), and (2.22), we have

$$\begin{aligned}\|\nabla(Tu)\|_{s,B,w} &\le C_{10}|B||B|^{-1/t}\|w\|_{1,B}^{1/s}|B|^{(m-t)/mt}\|u\|_{m,\rho B}\\ &\le C_{10}|B||B|^{-1/m}\|w\|_{1,B}^{1/s}\cdot\|1/w\|_{1/(r-1),\rho B}^{1/s}\|u\|_{s,\rho B,w} \quad (2.23)\\ &\le C_{11}|B|\|u\|_{s,\rho B,w}\end{aligned}$$

for all balls $B$ with $\rho B \subset \Omega$. Hence, (2.17) holds. The proof of Theorem 2.8 is completed.

Using a method similar to the proof of Theorem 2.8, we obtain

$$\|Tu\|_{s,B,w^\alpha} \le C|B|\operatorname{diam}(B)\|u\|_{s,\rho B,w^\alpha}, \tag{2.24}$$

where $\alpha$ is any real number with $0 < \alpha \le 1$ and $\rho > 1$.

Now we prove the following local weighted imbedding theorem for differential forms under the homotopy operator $T$.

THEOREM 2.25. *Let $u \in L^s_{\mathrm{loc}}(\Omega, \wedge^l)$, $l = 1, 2, \dots, n$, $1 < s < \infty$, be a differential form satisfying* (1.4) *in a bounded domain $\Omega \subset \mathbf{R}^n$ such that $du \in L^s_{\mathrm{loc}}(\Omega, \wedge^{l+1})$ and let $T$: $L^s(\Omega, \wedge^l) \to W^{1,s}(\Omega, \wedge^{l-1})$, $l = 1, 2, \dots, n$, be the operator defined in* (1.2). *Assume that $\rho > 1$ and $w \in A_r(\Omega)$ for some $r > 1$. Then, there exists a constant $C$, independent of $u$, such that*

$$\|Tu\|_{W^{1,s}(B),w^\alpha} \le C|B|\|u\|_{s,\rho B,w^\alpha} \tag{2.26}$$

*for all balls $B$ with $\rho B \subset \Omega$ and any real number $\alpha$ with $0 < \alpha \le 1$.*

*Proof.* From (1.1), (2.9)′, and (2.24), we have

$$\begin{aligned}\|Tu\|_{W^{1,s}(B),\,w^\alpha} &= \operatorname{diam}(B)^{-1}\|Tu\|_{s,B,w^\alpha} + \|\nabla(Tu)\|_{s,B,w^\alpha}\\ &\le \operatorname{diam}(B)^{-1}(C_1|B|\operatorname{diam}(B)\|u\|_{s,\rho B,w^\alpha}) + C_2|B|\|u\|_{s,\rho B,w^\alpha}\\ &\le C_1|B|\|u\|_{s,\rho B,w^\alpha} + C_2|B|\|u\|_{s,\rho B,w^\alpha}\\ &\le C_3|B|\|u\|_{s,\rho B,w^\alpha},\end{aligned}$$

which is equivalent to (2.26). The proof of Theorem 2.8 is completed.

Note that the parameter $\alpha$ in both Theorem 2.8 and Theorem 2.25 is any real number with $0 < \alpha \le 1$. Therefore, we can have different versions of the weighted imbedding inequality by choosing different values for $\alpha$. For example, set $t = 1 - \alpha$ in Theorem 2.8 and write $d\mu = w(x)dx$. Then, inequality (2.9) becomes

$$\left(\int_B |\nabla(Tu)|^s w^{-t} d\mu\right)^{1/s} \le C|B|\left(\int_{\rho B} |u|^s w^{-t} d\mu\right)^{1/s}. \tag{2.27}$$

If we choose $\alpha = 1/r$ in Theorem 2.8, then (2.9) reduces to

$$\left(\int_B |\nabla(Tu)|^s w^{1/r} dx\right)^{1/s} \le C|B|\left(\int_{\rho B} |u|^s w^{1/r} dx\right)^{1/s}. \tag{2.28}$$

If we choose $\alpha = 1/s$ in Theorem 2.8, then $0 < \alpha < 1$ since $1 < s < \infty$. Thus, (2.9) reduces to the following symmetric version:

$$\left(\int_B |\nabla(Tu)|^s w^{1/s} dx\right)^{1/s} \leq C|B| \left(\int_{\rho B} |u|^s w^{1/s} dx\right)^{1/s}. \tag{2.29}$$

Finally, if we choose $\alpha = 1$ in Theorem 2.8, we have the following weighted imbedding inequality:

$$\|\nabla(Tu)\|_{s,\,B,\,w} \leq C|B| \|u\|_{s,\,\rho B,\,w}. \tag{2.30}$$

*Remark.* Choosing $\alpha$ to be some special values in Theorem 2.25, we shall have some similar results. For example, selecting $\alpha = 1$ in Theorem 2.25, we have

$$\|Tu\|_{W^{1,s}(B),\ w} \leq C|B| \|u\|_{s,\,B,\,w}. \tag{2.31}$$

Considering the length of the paper, we do not list these similar results here.

## 3. THE GLOBAL WEIGHTED IMBEDDING THEOREM

We need the following properties of the Whitney covers appearing in [9] to prove the global result.

LEMMA 3.1. *Each $\Omega$ has a modified Whitney cover of cubes $\mathscr{V} = \{Q_i\}$ such that*

$$\cup_i Q_i = \Omega,$$

$$\sum_{Q \in \mathscr{V}} \chi_{(\sqrt{5/4})Q} \leq N\chi_\Omega$$

*for all $x \in \mathbf{R}^n$ and some $N > 1$, and if $Q_i \cap Q_j \neq \phi$, then there exists a cube $R$ (this cube does not need to be a member of $\mathscr{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover, if $\Omega$ is $\delta$-John, then there is a distinguished cube $Q_0 \in \mathscr{V}$ which can be connected with every cube $Q \in \mathscr{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathscr{V}$ and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.*

We prove the following global $A_r(D)$-weighted imbedding theorem in a bounded domain $D$ for differential forms.

THEOREM 3.2. *Let $u \in L^s(D, \wedge^l)$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, be a differential form satisfying* (1.4) *in a bounded, convex domain $D \subset \mathbf{R}^n$ and let*

$$T\colon L^s(D, \wedge^l) \to W^{1,s}(D, \wedge^{l-1}), \qquad l = 1, 2, \ldots, n,$$

*be a homotopy operator defined by* (1.3). *Assume that* $w \in A_r(D)$ *for some* $r > 1$. *Then, there exists a constant* $C$, *independent of* $u$, *such that*

$$\|\nabla(Tu)\|_{s,D,w^\alpha} \le C\|u\|_{s,D,w^\alpha}, \tag{3.3}$$

$$\|Tu\|_{W^{1,s}(D),w^\alpha} \le C\|u\|_{s,D,w^\alpha} \tag{3.4}$$

*for any real number* $\alpha$ *with* $0 < \alpha \le 1$.

*Proof.* Using (2.9) and Lemma 3.1, we find that

$$\begin{aligned}\|\nabla(Tu)\|_{s,D,w^\alpha} &= \left(\int_D |\nabla(Tu)|^s w^\alpha dx\right)^{1/s} \\ &\le \sum_{Q\in\mathcal{V}} \left(C_1|Q|\left(\int_{\rho Q} |u|^s w^\alpha dx\right)^{1/s}\right) \\ &\le C_1|D| \sum_{Q\in\mathcal{V}} \left(\int_{\rho Q} |u|^s w^\alpha dx\right)^{1/s} \\ &\le C_1|D| \sum_{Q\in\mathcal{V}} \left(\int_D |u|^s w^\alpha dx\right)^{1/s} \\ &\le C_3\left(\int_D |u|^s w^\alpha dx\right)^{1/s} \\ &= C_3\|u\|_{s,D,w^\alpha}\end{aligned} \tag{3.5}$$

since $D$ is bounded. We have proved that (3.3) holds. Similarly, using Lemma 3.1 and (2.24), we have

$$\|Tu\|_{s,D,w^\alpha} \le C_4 \operatorname{diam}(D)\|u\|_{s,D,w^\alpha}. \tag{3.6}$$

Now combining (1.1), (3.5), and (3.6) yields

$$\begin{aligned}\|Tu\|_{W^{1,s}(D),w^\alpha} &= \operatorname{diam}(D)^{-1}\|Tu\|_{s,D,w^\alpha} + \|\nabla(Tu)\|_{s,D,w^\alpha} \\ &\le C_4\|u\|_{s,D,w^\alpha} + C_3\|u\|_{s,D,w^\alpha} \\ &\le C_5\|u\|_{s,D,w^\alpha},\end{aligned}$$

which indicates that (3.4) holds. The proof of Theorem 3.2 has been completed.

*Remark.* Choosing $\alpha$ to be some special values in (3.3) and (3.4), we shall have some global results similar to the local case. For example, if let $\alpha = 1$, then (3.3) and (3.4) become

$$\|\nabla(Tu)\|_{s,D,w} \le C\|u\|_{s,D,w}, \tag{3.7}$$

$$\|Tu\|_{W^{1,s}(D),w} \le C\|u\|_{s,D,w}, \tag{3.8}$$

respectively. Considering the length of the paper, we leave it to the reader to find the similar global results.

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